Quasistatic magnetic field generated by a short laser pulse in an underdense plasma

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(Received 20 May 1997; accepted 22 August 1997)

The quasistatic magnetic field generated by short laser pulse in a uniform rarefied plasma is found analytically and compared to that of two-dimensional particle simulations. It is shown that an axisymmetric laser pulse generates an azimuthal magnetic field the structure of which is rather complicated inside the laser pulse body. Behind the pulse in the wake region the magnetic field contains a component that is homogeneous in the longitudinal direction and a component that is oscillating at the wave number $2k_p$, where k_p is the wave number of the plasma wake. Particle simulations confirm the analytical results and are also used to treat the case of high intense laser pulses. © *1997 American Institute of Physics.* [S1070-664X(97)00212-7]

I. INTRODUCTION

Recent progress in the generation of extremely intense short laser pulses (see, for example, Ref. 1) has stimulated a great interest in the problem of their interaction with matter. In particular, the self-generation of quasistatic magnetic fields is a subject of increasing attention. Using the term ''quasistatic'' we mean magnetic fields with slow variation on the time scale of the period of the laser radiation.

Recently, self-generated magnetic fields of a few hundred megagauss have been revealed in numerical simulations of ultraintense laser pulse interaction with an overdense nonuniform plasma.[2] The physical mechanism responsible for such great magnetic fields was discussed by Sudan.[3] In an underdense inhomogeneous plasma the nonrelativistic two-dimensional treatment of self-generated magnetic fields was presented by Tripathi and Liu.[4] They showed that an amplitude modulated laser beam propagating along a plasma density gradient produces a rotational current that gives rise to a quasistatic magnetic field. An analogous mechanism was considered earlier in the paper of Dysthe *et al.*,[5] where the nonlinear mixing of two electromagnetic waves in nonuniform plasma was examined. Another mechanism of magnetic field generation in a rarefied uniform plasma was discussed by Bychenkov *et al.*[6] They investigated a circularly polarized pulse for which the generation of a low-frequency electromagnetic field is due to the inverse Faraday effect. In the extreme relativistic regime the magnetic field generated by the laser beams in underdense plasma was studied numerically recently.[7]

The main objective of this work is to investigate self-generated quasistatic magnetic fields, both in the laser pulse body and behind the pulse in the region of the wakefield. We treat the laser radiation as linearly polarized and the plasma as uniform and underdense. Our analytical work is based on a perturbation theory applied to the set of relativistic cold electron fluid equations and Maxwell's equations. The self-generation of quasistatic magnetic fields takes place in fourth order with respect to parameter v_E/c, where v_E and c are the electron quiver velocity and the light velocity, respectively. We find that a linearly polarized laser pulse with an axisymmetric envelope generates an azimuthal magnetic field. The structure of this magnetic field inside the laser pulse body depends essentially on the pulse shape. In the wake region, in a frame moving with the pulse, the magnetic field contains a component that is homogeneous in the longitudinal direction and is due to the steady current produced by the plasma wake field, and a component that is oscillating at the wave number $2k_p$, where k_p is the wave number of the plasma wake, a known property of nonlinear plasma waves.[8,9] Numerical particle simulations confirm the analytical results and are also used to treat the case of high intense laser pulses with $v_E/c \gtrsim 1$.

II. BASIC EQUATIONS AND APPROXIMATIONS

We begin our considerations with the Maxwell's equations and the hydrodynamic equations for a relativistic cold electron fluid (see, for example, Ref. 8):

$$\nabla\times\mathbf{E}=-\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}, \tag{1}$$

$$\nabla\times\mathbf{B}=\frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}+\frac{4\pi e}{c}(n_0+n)\mathbf{v}, \tag{2}$$

$$\nabla\cdot\mathbf{E}=4\pi e n, \quad \nabla\cdot\mathbf{B}=0, \tag{3}$$

$$\frac{\partial\mathbf{p}}{\partial t}+(\mathbf{v}\cdot\nabla)\mathbf{p}=e\mathbf{E}+\frac{e}{c}(\mathbf{v}\times\mathbf{B}), \tag{4}$$

$$\frac{\partial n}{\partial t}+\nabla\cdot[(n+n_0)\mathbf{v}]=0, \tag{5}$$

$$\mathbf{p}=m\mathbf{v}\left(1-\frac{v^2}{c^2}\right)^{-1/2}, \tag{6}$$

where n_0 is the density of ions that are supposed to be immobile, single charged, and uniformly distributed in space, n is the difference between the ion and electron densities, $\mathbf{p}$ and $\mathbf{v}$ are the momentum and velocity of the electron fluid, respectively, and $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic fields.

The full system of Eqs. (1)–(6) can be reduced to one equation for the dimensionless electron fluid momentum $\mathbf{q}=\mathbf{p}/(mc)$.[10] To prove this statement we take the curl of Eq. (4) and use Eq. (1) and the expression

$$\mathbf{v}\times\nabla\times\mathbf{q}=c\nabla\sqrt{1+q^2}-(\mathbf{v}\cdot\nabla)\mathbf{q}.$$

As a result, we obtain the equation for the generalized vorticity $\Omega=c\nabla\times\mathbf{q}+e\mathbf{B}/(mc)$ in the following form:

$$\frac{\partial\Omega}{\partial t}=\nabla\times(\mathbf{v}\times\Omega). \tag{7}$$

According to (7) the flux of the generalized vorticity through an arbitrary surface bounded by a contour moving together with the fluid is constant (Kelvin's theorem[11]). Therefore, provided that the value Ω is zero initially before the pulse appears, it is zero for all subsequent times, and consequently

$$\frac{e\mathbf{B}}{mc}=-c\nabla\times\mathbf{q}. \tag{8}$$

Substituting Eq. (8) into Eq. (4), we obtain an expression for the electric field in the form

$$\frac{e\mathbf{B}}{mc}=e\nabla\sqrt{1+q^2}+\frac{\partial\mathbf{q}}{\partial t}. \tag{9}$$

Then, in accordance with Poisson's equation (3) we find the electron density,

$$\frac{n}{n_0}=\frac{c}{\omega_p^2}\nabla\cdot\left(\frac{\partial\mathbf{q}}{\partial t}+c\nabla\sqrt{1+q^2}\right), \tag{10}$$

where $\omega_p=\sqrt{4\pi e^2 n_0/m}$ is the electron plasma frequency determined by the ion density.

Taking the temporal derivative from Eq. (9) and using Eqs. (2), (8), and (10), we obtain an equation involving only the dimensionless electron momentum $\mathbf{q}$,

$$\frac{\partial^2\mathbf{q}}{\partial t^2}+c^2\nabla\times\nabla\times\mathbf{q}+\frac{\omega_p^2}{\sqrt{1+q^2}}$$
$$=-c\frac{\partial}{\partial t}\nabla\sqrt{1+q^2}-\frac{c\mathbf{q}}{\sqrt{1+q^2}}\nabla\cdot\left(\frac{\partial\mathbf{q}}{\partial t}+c\nabla\sqrt{1+q^2}\right). \tag{11}$$

In this paper we are interested in the weak nonlinear limit when the electron fluid velocity v_E is substantially less than the light velocity c. We then expand Eq. (11) with respect to parameter $q<1$ up to fourth-order terms and present the result in the form

$$\frac{\partial^2\mathbf{q}}{\partial t^2}+c^2\nabla\times\nabla\times\mathbf{q}+\omega_p^2\mathbf{q}$$
$$=-\frac{c}{2}\nabla\frac{\partial}{\partial t}\left(q^2-\frac{1}{4}q^4\right)+\frac{c^2}{2}\mathbf{q}\left(\frac{\omega_p^2}{c^2}-\Delta\right)q^2$$
$$-c\mathbf{q}\left(1-\frac{q^2}{2}\right)\nabla\cdot\frac{\partial\mathbf{q}}{\partial t}. \tag{12}$$

We examine the nonlinear equation (12) order by order in powers of the small parameter (v_E/c) by writing

$$\mathbf{q}=\mathbf{q}_1+\mathbf{q}_2+\mathbf{q}_3+\mathbf{q}_4, \tag{13}$$

where the value $\mathbf{q}_n$ determines the electron fluid momentum of the nth order.

It is evident that the first order of Eq. (12) corresponds to the well-known linear approximation,

$$\frac{\partial^2\mathbf{q}_1}{\partial t^2}+c^2\nabla\times\nabla\times\mathbf{q}_1+\omega_p^2\mathbf{q}_1=0. \tag{14}$$

In accordance with Helmholtz's theorem,[12] the vector $\mathbf{q}_1$ may be decomposed into two parts $\mathbf{q}_1=\mathbf{q}_1^{\mathrm{long}}+\mathbf{q}_1^{\mathrm{tr}}$, such that $\nabla\cdot\mathbf{q}_1^{\mathrm{tr}}=0$ and $\nabla\times\mathbf{q}_1^{\mathrm{long}}=0$. For the rotational part $\mathbf{q}_1^{\mathrm{tr}}$ Eq. (14) has the form

$$\frac{\partial^2\mathbf{q}_1}{\partial t^2}-c^2\Delta\mathbf{q}_1^{\mathrm{tr}}+\omega_p^2\mathbf{q}_1^{\mathrm{tr}}=0. \tag{15}$$

The irrotational curl-free part $\mathbf{q}_1^{\mathrm{long}}$ satisfies to the equation

$$\frac{\partial^2\mathbf{q}_1^{\mathrm{long}}}{\partial t^2}+\omega_p^2\mathbf{q}_1^{\mathrm{long}}=0. \tag{16}$$

In particular, Eqs. (15) and (16) describe the transverse and longitudinal waves in a plasma, which in the linear approximation are uncoupled and propagate independently of one and another.

In the linear approximation, a laser pulse propagating in a homogeneous unmagnetized plasma is a transverse electromagnetic wave. Therefore, we are interested in rotational first-order plasma motion with $\nabla\cdot\mathbf{q}_1=0$. To simplify the notation we omit the superscript ''tr'' below.

Keeping in mind the condition $\nabla\cdot\mathbf{q}_1=0$, we obtain from (12) the equation for $\mathbf{q}_2$ in the form

$$\frac{\partial^2\mathbf{q}_2}{\partial t^2}+c^2\nabla\times\nabla\times\mathbf{q}_2+\omega_p^2\mathbf{q}_2=-\frac{c}{2}\nabla\frac{\partial}{\partial t}q_1^2. \tag{17}$$

The gradient form of the driving term in the right side of Eq. (17) imposes the condition $\nabla\times\mathbf{q}_2=0$. This means that the second-order magnetic field [see Eq. (8)] is absent. Thus, in homogeneous collisionless plasmas the solenoidal electromagnetic wave does not generate a magnetic field in second order. Second-order dc magnetic fields can be generated by factors that are not present for short laser pulses and are ignored in our analysis (e.g., by collisions of electrons,[13,14] plasma inhomogeneity,[4,5,15] irrotational form of the zeroth-order electric field[6,14]).

The solution of Eq. (17) satisfying the absence of perturbations in front of the pulse has the form

$$\mathbf{q}_2(\mathbf{r},t)=-\frac{c}{2\omega p}\,\boldsymbol{\nabla}\int_{-\infty}^{t}dt'\,\sin[\omega_p(t-t')]\,\frac{\partial}{\partial t'}\,q_1^2(\mathbf{r},t'). \tag{18}$$

In accordance with (12), the equation for the third-order value $\mathbf{q}_3$ has the form

$$\frac{\partial^2\mathbf{q}_3}{\partial t^2}+c^2\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\mathbf{q}_3+\omega_p^2\mathbf{q}_3 =-c\boldsymbol{\nabla}\,\frac{\partial}{\partial t}\,(\mathbf{q}_1\cdot\mathbf{q}_2)+\mathbf{q}_1\left(\frac{1}{2}\,(\omega_p^2-c^2\Delta)q_1^2-c\boldsymbol{\nabla}\cdot\,\frac{\partial\mathbf{q}_2}{\partial t}\right). \tag{19}$$

This equation may be used to study third harmonic generation. Recently, this problem was discussed for short laser pulses in several papers.[16] In addition, Eq. (19) determines the nonlinear second-order frequency shift in the dispersion relation of the laser radiation.

In our aim to investigate the magnetic field, we need to know $\boldsymbol{\nabla}\cdot\mathbf{q}_3$. From Eq. (19) we obtain for this quantity the following equation:

$$\left(\frac{\partial^2}{\partial t^2}+\omega_p^2\right)\boldsymbol{\nabla}\cdot\mathbf{q}_3=\mathbf{q}_1\cdot\boldsymbol{\nabla}\left(\frac{1}{2}\,(\omega_p^2-c^2\Delta)q_1^2-c\boldsymbol{\nabla}\cdot\,\frac{\partial\mathbf{q}_2}{\partial t}\right) -c\,\frac{\partial}{\partial t}\,\Delta(\mathbf{q}_1\cdot\mathbf{q}_2). \tag{20}$$

The solution of Eq. (20) is quite similar to Eq. (18) and has the form

$$\boldsymbol{\nabla}\cdot\mathbf{q}_3=\frac{1}{\omega_p}\int_{-\infty}^{1}dt'\,\sin[\omega_p(t-t')]\left\{\mathbf{q}_1\boldsymbol{\nabla}\left[\frac{1}{2}\,(\omega_p^2 -c^2\Delta)q_1^2-c\boldsymbol{\nabla}\cdot\,\frac{\partial\mathbf{q}_2}{\partial t}\right]-c\,\frac{\partial}{\partial t'}\,\Delta(\mathbf{q}_1\cdot\mathbf{q}_2)\right\}, \tag{21}$$

where the values $\mathbf{q}_1$ and $\mathbf{q}_2$ have to be evaluated at time t'.

The equation for $\mathbf{q}_4$, being analogous to Eq. (19), describes the generation of the second, third, and fourth harmonic radiation as well as the excitation of the fourth-order quasisteady magnetic field. Because the last problem is the focus of our interest, we present here only the equation for $\mathbf{B}_4=-(mc^2/e)\mathrm{rot}\,\mathbf{q}_4$,

$$\frac{\partial^2\mathbf{B}}{\partial t^2}-c^2\Delta\mathbf{B}+\omega_p^2\mathbf{B}=4\pi c\boldsymbol{\nabla}\times(\mathbf{j}_1+\mathbf{j}_2), \tag{22}$$

where we have omitted the subscript in the magnetic field ($\mathbf{B}=\mathbf{B}_4$) and decomposed the fourth-order current into two components, one of which is proportional to the vector $\mathbf{q}_2$ and has the form

$$\mathbf{j}_1=-\frac{mc}{4\pi e}\,\mathbf{q}_2\left(\frac{1}{2}\,(\omega_p^2-c^2\Delta)q_1^2-c\,\frac{\partial}{\partial t}\,\boldsymbol{\nabla}\cdot\mathbf{q}_2\right). \tag{23}$$

The other component is proportional to the vector $\mathbf{q}_1$ and has the form

$$\mathbf{j}_2=-\frac{mc}{4\pi e}\,\mathbf{q}_1\left((\omega_p^2-c^2\Delta)(\mathbf{q}_1\cdot\mathbf{q}_2)-c\,\frac{\partial}{\partial t}\,\boldsymbol{\nabla}\cdot\mathbf{q}_3\right). \tag{24}$$

Equations (23) and (24) can be transformed to forms that are more convenient for analysis. For this reason we introduce the dimensionless function

$$\phi(\mathbf{r},t)=\omega_p\int_{-\infty}^{t}dt'\,\sin[\omega_p(t-t')]q_1^2(t',\mathbf{r}), \tag{25}$$

which is the solution of the equation

$$\frac{\partial^2\phi}{\partial t^2}+\omega_p^2\phi=\omega_p^2q_1^2.$$

Then, in accordance with Eq. (18), we have

$$\mathbf{q}_2=-\frac{c}{2\omega_p^2}\,\boldsymbol{\nabla}\,\frac{\partial\phi}{\partial t}. \tag{26}$$

Substituting (25) and (26) into Eqs. (23) and (24), we obtain, after a few simple manipulations,

$$\mathbf{j}_1=\frac{ec^2n_0}{4\omega_p^2}\left(\boldsymbol{\nabla}\,\frac{\partial\phi}{\partial t}\right)\left(q_1^2-\frac{c^2}{\omega_p^2}\,\Delta\phi\right), \tag{27}$$

and

$$\mathbf{j}_2=-ecn_0\mathbf{q}_1\left\{(\mathbf{q}_1\cdot\mathbf{q}_2)-\frac{c^2}{\omega_p}\,\Delta\int_{-\infty}^{t}dt'\,\sin[\omega_p(t-t')]\times[\mathbf{q}_1(t')\cdot\mathbf{q}_2(t')]-\frac{c}{2}\int_{-\infty}^{t}dt'\,\cos[\omega_p(t-t')]\times\left[\mathbf{q}_1(t')\cdot\boldsymbol{\nabla}\left(q_1^2(t')-\frac{c^2}{\omega_p^2}\,\Delta\phi(t')\right)\right]\right\}. \tag{28}$$

III. QUASISTEADY NONLINEAR CURRENT

For the first-order function $\mathbf{q}_1$ determining the form of the laser pulse in the linear approximation, we assume

$$\mathbf{q}_1(\mathbf{r},t)=\tfrac{1}{2}[\mathbf{a}(\mathbf{r},t)\exp(-i\omega t+ikz)+\text{c.c.}], \tag{29}$$

where c.c. denotes the complex conjugate. The laser pulse is taken to be propagating along the z axis in the positive direction, ω and k are the frequency and wave number that are related by the dispersion law $k^2c^2=\omega^2-\omega_p^2$, $\mathbf{a}$ is the complex amplitude that is supposed to be a slowly varying function both in time on the laser radiation period $(2\pi/\omega)$ and in space on its wavelength $(2\pi/k)$.

In the lowest approximation with respect to spatial and temporal derivatives of the amplitude $\mathbf{a}$, we obtain from Eq. (15),

$$\frac{\partial\mathbf{a}}{\partial t}+v_g\,\frac{\partial\mathbf{a}}{\partial z}=0, \tag{30}$$

where $v_g=c^2k/\omega$ is the group velocity of the pulse. The solution of Eq. (30) is an arbitrary function of the variable $\xi=v_gt-z$, representing the axial coordinate in a frame moving together with the laser pulse, and of the variable $\mathbf{r}_\perp$, determining the coordinate in the plane transverse to the direction of propagation of the laser pulse. In this paper we consider the nonlinear plasma response to a pulse of a given unchanged form. Hence, we omit the effects of the diffraction and dispersion, arising as a result of the second spatial and temporal derivatives of the slowly changing amplitude $\mathbf{a}$.

We also ignore the influence of nonlinear effects on a pulse. Our justification for this is that the neglected effects cause the pulse envelope to change slowly. Our calculation of the magnetic field will thus apply to the case of a pulse with specified basic parameters, e.g. width, duration, amplitude.

Using Eq. (29), we find that the function q_1^2 has the form

$$q_1^2=\tfrac{1}{2}|a|^2+\tfrac{1}{4}[a^2\exp(-2i\omega t+2ikz)+\text{c.c.}], \tag{31}$$

where the first term varies slowly both in space and in time as opposed to the second one containing rapidly varying exponential factors (second harmonics). As a consequence of Eq. (31) the function ϕ, defined by Eq. (25), contains also two terms ($\phi=\phi_0+\phi_2$) with the analogous properties. The slowly varying component ϕ_0 has the form

$$\phi_0=\frac{\omega_p}{2}\int_{-\infty}^{t}dt'\,\sin[\omega_p(t-t')]|a(\mathbf{r},t')|^2, \tag{32}$$

while the second harmonic term ϕ_2 is given by

$$\phi_2=\frac{\omega_p}{4}\int_{-\infty}^{t}dt'\,\sin[\omega_p(t-t')][a^2(t',\mathbf{r})\exp(-2i\omega t'+2ikz)+\text{c.c.}]. \tag{33}$$

The integrals in Eq. (33) contain the product of functions with quite different temporal dependences. The factors exp $(\pm 2i\omega t')$ are changing significantly faster than the other functions. Hence, after repeated integration by parts, we are able to find the resultant ϕ_2 in the form of a power series in ω^{-1}. The general expression of this procedure has the form

$$\int_{-\infty}^{t}dt'\,F(t')e^{-i\omega t'}=\frac{i}{\omega}e^{-i\omega t}\left[F(t)+\sum_{n=1}^{\infty}\left(-\frac{i}{\omega}\right)^n\left(\frac{\partial^n F(t')}{\partial t'^n}\right)_{t'=t}\right], \tag{34}$$

where $F(t)$ is supposed to be some slowly varying function in time on the scale ω^{-1}.

By means of (34) we can calculate ϕ_2 to any accuracy with respect to the operator $\partial/(\omega\partial t)$. Further, substituting the function ϕ in Eq. (26) and taking into account the operations

$$\boldsymbol{\nabla}(Fe^{ikz})=e^{ikz}(ik\mathbf{e}_z+\boldsymbol{\nabla})F, \tag{35}$$

and

$$\Delta(Fe^{ikz})=e^{ikz}\left(-k^2+2ik\frac{\partial}{\partial z}+\Delta\right)F, \tag{36}$$

we can find the function $\mathbf{q}_2$. Using the expressions obtained for ϕ and $\mathbf{q}_2$ and also Eqs. (34)–(36) we are able to calculate the currents (27) and (28), including all their harmonics. Our interest is restricted here to the quasisteady current (zero harmonic). Further, we limit ourself to the second-order terms with respect to the small parameters, $\partial/(\omega\partial t)$, ω_p/ω, $\partial/(k\partial z)$, and $k^{-1}\boldsymbol{\nabla}_\perp$, where the operator $\boldsymbol{\nabla}_\perp$ means the spatial gradient in the transverse direction.

On the basis of calculations presented in Appendices A and B, we conclude that the main contribution to the quasisteady current comes from the slowly varying quantity ϕ_0, and thus,

$$\overline{\mathbf{J}}=\frac{ec^2n_0}{4\omega_p^2}\left(\boldsymbol{\nabla}\frac{\partial\phi_0}{\partial t}\right)\left(\frac{|a|^2}{2}-\frac{c^2}{\omega_p^2}\Delta\phi_0\right), \tag{37}$$

where the overbar indicates that this current is a slowly varying one. All other terms in the current $\overline{\mathbf{J}}$ contain additional factors proportional to the square of the small parameters indicated above [see Eqs. (A7) and (A20)].

Behind the pulse, the function (32) determines the plasma wake excited by a pulse.[17] In the wake itself, Eq. (37) reduces to

$$\overline{\mathbf{J}}=-\frac{ec^4n_0}{4\omega_p^4}\left(\boldsymbol{\nabla}\frac{\partial\phi_0}{\partial t}\right)\Delta\phi_0. \tag{38}$$

The current (38) may be represented in another form, $\mathbf{j}=en_1\mathbf{v}_1$, where n_1 and $\mathbf{v}_1$ are the electron density and velocity perturbations in the plasma wake. Therefore, in accordance with Eq. (22), the quasistatic part of magnetic field $\mathbf{B}$ is determined by the equation (see also Ref. 8)

$$\left(\frac{\partial^2}{\partial t^2}-c^2\Delta+\omega_p^2\right)\mathbf{B}=4\pi ec\boldsymbol{\nabla}n_1\times\mathbf{v}_1. \tag{39}$$

The origin of the magnetic field in Eq. (39) has a form analogous to the one considered in Refs. 4, 5, where an inhomogeneous plasma was investigated. However, in these papers the background plasma inhomogeity produces the electron density gradient, whereas in our case the density perturbations are produced by the plasma wave.

IV. QUASISTATIC MAGNETIC FIELD: GENERAL RELATIONS

We now specialize our considerations to a steady axisymmetric laser pulse. Introducing the new variable $\xi=v_gt-z$, we rewrite Eq. (32) in the form

$$\phi_0(r,\xi)=\frac{k_p}{2}\int_{-\infty}^{\xi}d\xi'\,\sin[k_p(\xi-\xi')]|a(\xi',r)|^2, \tag{40}$$

where $k_p=(\omega_p/v_g)$ is the plasma wave number, r is the radial variable. In this case the current (37) contains only axial and radial components, which, under the condition $v_g=c$, are

$$\overline{j_z}=-\frac{ecn_0}{4k_p^4}\frac{\partial^2\phi_0}{\partial\xi^2}[(k_p^2-\Delta_\perp)\phi_0], \tag{41}$$

$$\overline{j_r}=\frac{ecn_0}{4k_p^4}\frac{\partial^2\phi_0}{\partial r\partial\xi}[(k_p^2-\Delta_\perp)\phi_0], \tag{42}$$

where $\Delta_\perp=r^{-1}(\partial/\partial r)(r\partial/\partial r)$ is the transverse part of the Laplace operator.

As it turns out, only an azimuthal component of the magnetic field $B_\varphi=B$ exists in our case for which, in accordance with Eqs. (22), (37), (41), and (42), we obtain the equation

$$\left(\Delta_\rho-\frac{1}{\rho^2}-1\right)B=\frac{\pi e n_0}{k_p^3}F(\xi,\rho), \tag{43}$$

where $\rho=k_p r$ is a dimensionless radial variable, $\Delta_\rho=\rho^{-1}(\partial/\partial\rho)(\rho\partial/\partial\rho)$, and

$$F(\xi,\rho)=\frac{\partial^2\phi_0}{\partial\xi\partial\rho}\left((1-\Delta_\rho)\frac{\partial\phi_0}{\partial\xi}\right)-\frac{\partial^2\phi_0}{\partial\xi^2}\frac{\partial}{\partial\rho}\times[(1-\Delta_\rho)\phi_0]. \tag{44}$$

The solution of Eq. (44), satisfying boundary conditions $B(\rho=0)=B(\rho\to\infty)=0$, is given by

$$B=\frac{\pi e n_0}{k_p^3}\left(I_1(\rho)\int_\rho^\infty x\,dx\,K_1(x)F(\xi,x)+K_1(\rho)\int_0^\rho x\,dx\,I_1(x)F(\xi,x)\right), \tag{45}$$

where I_1 and K_1 are modified Bessel functions of the first and second kinds, respectively.[18]

It is seen from (45) that the axial dependence of the magnetic field is determined by the function F. In turn, the function F depends on the function ϕ_0, which is determined by the axial form of the pulse. Therefore, the axial variation of the magnetic field B inside a pulse depends essentially on its form. However, behind the pulse, where the function (40) determines the plasma wake excited by a pulse,[17] the axial variation of the magnetic field is universal.

To prove this statement we suppose that the back pulse boundary is situated by $\xi=L$. Then, behind the pulse, $\xi>L$, the function (40) has the form

$$\phi_0=\psi\sin(k_p\xi-\theta), \tag{46}$$

where ψ and θ are the amplitude and phase of the plasma wakefield, respectively,

$$\psi=\sqrt{\psi_1^2+\psi_2^2},\quad \tan\theta=(\psi_1/\psi_2), \tag{47}$$

where ψ_1 and ψ_2 are determined by the longitudinal pulse profile

$$\psi_1=\frac{k_p}{2}\int_{-\infty}^{L}d\xi'\,\sin(k_p\xi')|a(\xi',\rho)|^2,$$
$$\psi_2=\frac{k_p}{2}\int_{-\infty}^{L}d\xi'\,\cos(k_p\xi')|a(\xi',\rho)|^2. \tag{48}$$

Substituting (46) in (44) we find, after some manipulations,

$$F(\rho,\xi)=\frac{k_p^2}{2}\{[\psi^2-\psi\,\Delta_\rho\psi+\psi^2\theta'^2]'+\cos[2(k_p\xi-\theta)]\times[\psi(\Delta_\rho\psi)'-(\Delta_\rho\psi)\psi'+\tfrac{1}{2}(\psi^2)'(\theta')^2-\psi(\psi\theta'^2)']+\sin[2(k_p\xi-\theta)][2\psi(\psi'\theta')'-2\theta'(\psi')^2+\psi(\Delta_\rho\theta)']\}, \tag{49}$$

where overscript$'$ denotes the differentiation on ρ.

Therefore, in the frame moving with the pulse, the magnetic field, in the region of the plasma wakefield, contains a component that is homogeneous in the longitudinal direction and a component that is oscillating at the wave number $2k_p$, where $k_p=\omega_p/c$ is the wave number of the plasma wake. This same conclusion was obtained in Refs. 8, 9 as a property of a nonlinear plasma wave.

The magnetic field (45) may be substantially simplified for a laser pulse of the form

$$|a(\xi,r)|^2=a_0^2 f_1(\xi)f_2(r), \tag{50}$$

where the value $a_0^2=(|v_{E,\max}^2|/c^2)$ determines the peak normalized radiation intensity in the pulse. The functions f_1 and f_2 are normalized to 1 and determine the pulse profiles in the longitudinal and transverse directions, respectively. Substituting (50) into Eq. (40) and then the result in Eq. (44), we obtain

$$F(\xi,\rho)=a_0^4\left[\left(\frac{d\varphi}{d\xi}\right)^2 g_1-\varphi\frac{d^2\varphi}{d\xi^2}g_2\right], \tag{51}$$

where

$$\varphi(\xi)=\frac{k_p}{2}\int_{-\infty}^{\xi}d\xi'\,\sin[k_p(\xi-\xi')]f_1(\xi') \tag{52}$$

$$g_1=\frac{df_2}{d\rho}[(1-\Delta_\rho)f_2],\quad g_2=f_2\frac{d}{d\rho}[(1-\Delta_\rho)f_2]. \tag{53}$$

Inserting (51) into Eq. (43), we find the magnetic field as a difference of two terms:

$$B=\frac{\pi e n_0}{k_p^3}a_0^4\left[\left(\frac{d\varphi}{d\xi}\right)^2 G_1(\rho)-\left(\varphi\frac{d^2\varphi}{d\xi^2}\right)G_2(\rho)\right], \tag{54}$$

where

$$G_{1,2}(\rho)=I_1(\rho)\int_\rho^\infty x\,dx\,K_1(x)g_{1,2}(x)+K_1(\rho)\int_0^\rho x\,dx\,I_1(x)g_{1,2}(x). \tag{55}$$

Every term in Eq. (54) contains the product of two factors. Each of them depends only on one variable and may be calculated by means of Eqs. (52) and (55).

Behind a pulse of the form (50), expression (49) and, consequently, the magnetic field (45) may be also simplified considerably. In this case, according to (48), $\Psi_{1,2}=a_0^2 f_2(r)\zeta_{1,2}(\zeta)$, where

$$\zeta_1=\frac{k_p}{2}\int_{-\infty}^{L}d\xi'\,\sin(k_p\xi')f_1(\xi'),$$
$$\zeta_2=\frac{k_p}{2}\int_{-\infty}^{L}d\xi'\,\cos(k_p\xi')f_1(\xi'). \tag{56}$$

From Eq. (47), it follows that

$$\psi=a_0^2 f_2(r)\zeta,\quad \tan\theta=\zeta_1/\zeta_2, \tag{57}$$

where $\zeta=\sqrt{\zeta_1^2+\zeta_2^2}$. Taking in mind that in the case under consideration $d\theta/d\rho=0$, we find by means of Eqs. (49) and (45) the magnetic field behind the pulse,

$$B=\pi e n_0 \frac{a_0^4\ \zeta^2}{2k_p}\{(G_1+G_2)+\cos[2(k_p\xi-\theta)]\times(G_1-G_2)\}. \tag{58}$$

So, to find the magnetic field in an evident form we have to evaluate the functions f_1 and f_2, to calculate the integrals (55) and (52) [or (56)] to make the operations indicated in Eq. (54).

V. QUASISTATIC MAGNETIC FIELD: EXPLICIT ANALYTICAL RESULTS

In this section we consider the example of a pulse with a form allowing explicit analytical results for the magnetic field. The longitudinal envelope of the pulse is taken to be of the same form as in our numerical simulations,

$$f_1(\xi)=\sin^2(\pi\xi/L), \tag{59}$$

where we assume that the pulse is situated in the region $0<\xi<L$ and the value L determines the pulse length.

Substituting (59) into Eq. (52), we find

$$\varphi=\frac{1}{2(k_L^2-k_p^2)}\left[k_L^2\sin^2\left(\frac{k_p\xi}{2}\right)-k_p^2\sin^2\left(\frac{k_L\xi}{2}\right)\right], \tag{60}$$

where $k_L=2\pi/L$. By means of Eq. (60) we are able to determine both those factors in Eq. (54), depending on the longitudinal pulse form.

In the transverse direction we assume the pulse shape to be Gaussian,

$$f_2(\rho)=\exp(-\alpha^2\rho^2), \tag{61}$$

where $\alpha^2=2(k_p r_L)^{-2}$, and r_L is the effective width of the pulse. Substituting (61) in (53), we obtain

$$g_1=-2\alpha^2\rho(1+4\alpha^2-4\alpha^4\rho^2)\exp(-2\alpha^2\rho^2), \tag{62}$$

$$g_2=-2\alpha^2\rho(1+8\alpha^2-4\alpha^4\rho^2)\exp(-2\alpha^2\rho^2). \tag{63}$$

In accordance with Eq. (55), the functions $G_{1,2}(\rho)$ have an integral representation. Simple explicit forms of these may be obtained both for narrow ($\alpha^2>1$) and for wide ($\alpha^2<1$) pulses.

In the case of a narrow pulse the functions $g_{1,2}$ are small for $\rho\geqslant 1$ as the factor $\exp(-2\alpha^2\rho^2)$ is small. Therefore, the region $\rho\leqslant 1$ gives the main contribution in integrals (55), where the simple expressions,

$$I_1\approx\frac{\rho}{2},\quad K_1\approx\frac{1}{\rho}, \tag{64}$$

are valid for the Bessel functions.[18] As a result, for $\alpha^2>1$, the functions $G_{1,2}$ are given by

$$G_1(\rho)=\frac{1}{8\alpha^2\rho}(e^{-2\alpha^2\rho^2}-\rho K_1)-\frac{\alpha^2\rho}{2}e^{-2\alpha^2\rho^2}, \tag{65}$$

$$G_2(\rho)=\frac{(1+4\alpha^2)}{8\alpha^2\rho}(e^{-2\alpha^2\rho^2}-\rho K_1)-\frac{\alpha^2\rho}{2}e^{-2\alpha^2\rho^2}. \tag{66}$$

In accordance with Eqs. (60) (65), and (66) the dimensionless magnetic field inside the body of a narrow pulse has the form

$$\frac{eB}{mc\omega p}=-\frac{a_0^4}{32[1-(k_\rho^2/k_L^2)]^2}\left\{\alpha^2\rho e^{-2\alpha^2\rho^2}\left[\sin\left(\frac{k_p\xi}{2}\right)\cos\left(\frac{k_L\xi}{2}\right)-\frac{k_p}{k_L}\sin\left(\frac{k_L\xi}{2}\right)\cos\left(\frac{k_p\xi}{2}\right)\right]^2+\frac{1}{\rho}(\rho K_1-e^{-2\alpha^2\rho^2})\times\left[\frac{k_p^2}{k_L^2}\sin^2\left(\frac{k_L\xi}{2}\right)-\sin^2\left(\frac{k_p\xi}{2}\right)\right]\left[\sin^2\left(\frac{k_L\xi}{2}\right)-\sin^2\left(\frac{k_p\xi}{2}\right)\right]\right\}. \tag{67}$$

Behind the narrow pulse in the wake region, we find, by means of Eqs. (56), (57), (59), (65), and (66), the magnetic field in the form

$$\frac{eB}{mc\omega_p}=-\frac{a_0^4\zeta^2}{16}\left(2\alpha^2\rho e^{-2\alpha^2\rho^2}+\frac{1}{\rho}(\rho K_1-e^{-2\alpha^2\rho^2})\times\{1-\cos[2(k_p\xi-\theta)]\}\right), \tag{68}$$

where

$$\zeta^2=\frac{\sin^2(k_pL/2)}{4[1-(k_p/k_L)^2]^2},\quad \theta=\frac{k_pL}{2}. \tag{69}$$

Equation (68) was obtained earlier.[19] As can be seen, Eqs. (67) and (68) coincide at the pulse boundary where $\xi=L$.

In the case of a wide laser pulse ($\alpha^2<1$), the integrals (55) can also be evaluated and the functions G_1 and G_2, as it is shown in Appendix C, have the form

$$G_1=-2\alpha^2\rho(1-12\alpha^2+12\alpha^4\rho^2)e^{-2\alpha^2\rho^2},\quad G_2=-2\alpha^2\rho(1-8\alpha^2+12\alpha^4\rho^2)e^{-2\alpha^2\rho^2}, \tag{70}$$

valid to terms of α^4 order. Then, the leading term in α^2 determining the magnetic field (54) inside the pulse body can be written as

$$\frac{eB}{mc\omega_p}=-\frac{a_0^4\alpha^2\rho\exp(-2\alpha^2\rho^2)}{8[1-(k_p/k_L)^2]^2}\left[\sin\left(\frac{k_p\xi}{2}\right)\cos\left(\frac{k_L\xi}{2}\right)-\frac{k_p}{k_L}\sin\left(\frac{k_L\xi}{2}\right)\cos\left(\frac{k_L\xi}{2}\right)\right]^2. \tag{71}$$

Inserting (70) into Eq. (58), we obtain the magnetic field behind the pulse,[19]

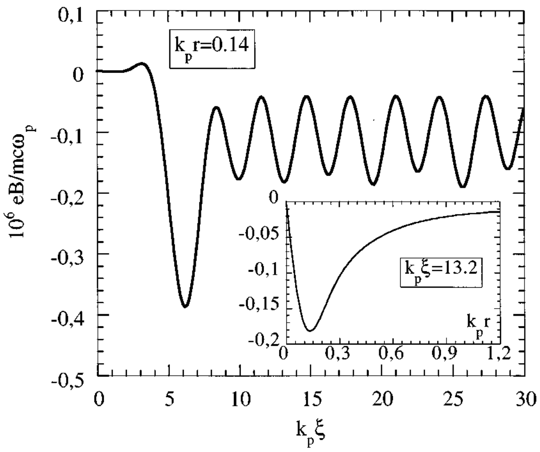

FIG. 1. Magnetic field generated inside a laser pulse ($k_p\xi<10$) and in the laser wake ($k_p\xi>10$) for a narrow laser pulse ($k_pr_L=0.3$) and a relatively low intensity ($a_0=0.04$). The plot shows the normalized magnetic field versus the dimensionless longitudinal variable taken at the distance $k_pr=\rho=0.14$ from the pulse axis, where the amplitude of the magnetic field is maximum in the wake region. The inset illustrates the radial form of the magnetic fields for $k_p\xi=13.2$.

$$\frac{eB}{mc\omega_p}=-\frac{a_0^4\zeta^2}{2}\,\alpha^2\rho e^{-2\alpha^2\rho^2}\{1-10\alpha^2+12\alpha^4\rho^2 -2\alpha^2\cos[2(k_p\xi-\theta)]\}, \tag{72}$$

where the values ζ^2 and θ are determined by Eq. (69) for the pulse shape considered here, and where we have kept terms up to α^4 order. It is can be seen that the oscillating part of the magnetic field (72) is proportional to the small factor α^2 compared to the main term. Therefore, a wide laser pulse generates mainly a homogeneous magnetic field in its wake.

VI. SIMULATIONS

A new fully nonlinear, relativistic, two-dimensional particle code WAKE,[20] which has been developed to simulate laser pulse propagation in an underdense plasma for high values of a_0, was applied to calculate the magnetic fields generated both inside the pulse body and behind it in the wake region. The longitudinal envelope of the pulse was taken of the form (59), where the value L was assumed to be $L=10k_p^{-1}$ or $k_L=0.628k_p$ henceforth.

The results for a narrow laser pulse with $\alpha^2=22.22$ ($k_pr_L=0.3$) and with relatively low intensity ($a_0=0.04$) are shown in Fig. 1. The plot shows the magnetic field versus $k_p\xi$ taken for radial variable $\rho=0.14$, where it has the maximum magnitude (see inset in Fig. 1) in the wake region. It is seen that, in accordance with the analytical prediction, in the wake region there is a homogeneous azimuthal magnetic field as well as an oscillating one with wave vector $2k_p$, that is, with wavelength $\lambda_p/2$, where λ_p is the plasma wavelength.

In order to make a quantitative comparison between simulation and analytical theory we have used Eqs. (67) and (68) to calculate the magnetic field and verified that the agreement with the simulation results is very good.

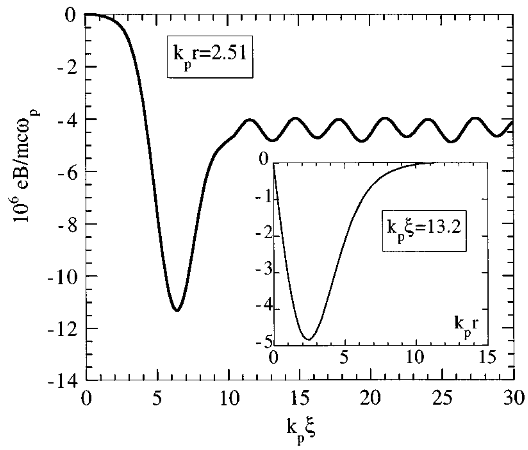

FIG. 2. The magnetic field for a wide laser pulse ($k_pr_L=6.28$, $\alpha^2=5.07\times10^{-2}$, $a_0=0.2$).

If we increase the laser intensity in this narrow pulse case, we observe that the agreement between the theoretical predictions and the simulation results is broken for a_0 of the order of 0.2, for which radial breaking of the plasma oscillations occurs.

Figure 2 shows the magnetic field of a wide ($\alpha^2=5.07\ 10^{-2}$ or $k_pr_L=2\pi$) and more intense ($a_0=0.2$) pulse. The parameters are otherwise identical to those shown in Fig. 1. We again obtain good agreement between the simulation results with the analytical ones. In accordance with Eq. (72), the amplitude of oscillating part of magnetic field is equal to $2\alpha^2$. For the parameters under consideration this amplitude is approximately 0.1 of the homogeneous part of magnetic field in accordance with the simulation result.

The numerical code, however, allows simulation of more intense laser pulses. Figure 3 illustrates the normalized mag-

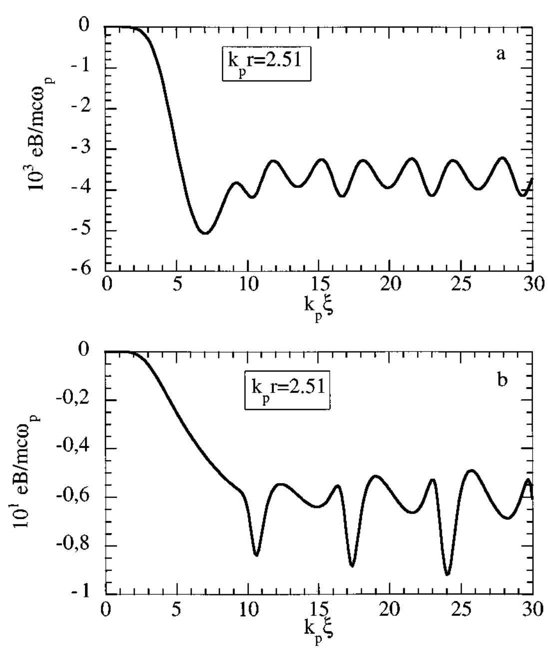

FIG. 3. The magnetic field for a wide laser pulse ($k_pr_L=6.28$) and for (a) $a_0=1$ and (b) $a_0=2$.

netic field as a function of the longitudinal coordinate for two pulses with the same width ($k_p r_L = 2\pi$ or $\alpha^2 = 5.07 \times 10^{-2}$), but with different dimensionless peak amplitudes a_0. For $a_0 = 1$, Fig. 3(a) shows that the magnetic field has a form analogous to that in the case $a_0 < 1$ (Fig. 2). The agreement between theory and simulation is still surprisingly good here, since using (72) beyond the limit of its validity for $a_0 = 1$ we obtain for the maximum value of the normalized homogeneous magnetic field behind the pulse $eB/mc\omega_p = -3.3\times 10^{-3}$ while the simulation gives -3.6×10^{-3}.

For a peak amplitude $a_0 = 2$, the simulation in Fig. 3(b) differs from the results of the analytical theory in the sense that the magnetic field oscillations in the wave region have no harmonic form and their periodicity does not correspond to $2k_p$. We think these effects reflect the nonlinear form of the wakefield generated by such an intense laser pulse. We verified, however, that in the frame of the moving pulse the electron flow in this case is still laminar in contrast with the narrow pulse case for $a_0 > 0.2$.

VII. CONCLUSION

In this paper, we have presented a theory and simulations of the quasistatic magnetic field generated by a short laser pulse in a rarefied plasma. On the basis of the relativistic electron fluid equations and the Maxwell equations, we found, in the fourth order of a perturbation theory, the quasistatic current and solved the equation describing the self-generated magnetic fields. In the limit of small ratio (v_E/c) (v_E and c are the electron quiver velocity and light velocity, respectively) the analytical results are in very good agreement with the results of two-dimensional particle simulations. Numerical simulations were used also to treat the case of intense laser pulses with $(v_E/c) \geqslant 1$.

It is interesting to pay attention to the physical mechanism responsible for generation of the homogeneous component of magnetic field in the region of a plasma wake. As it is seen from Eq. (39) this component of magnetic field is generated by the curl of the second-order averaged current produced by the plasma wake. The problem of the steady nonlinear current driven by a longitudinal plasma wave is an old one (see, e.g., Ref. 21 and the literature cited there).

It is well known that in the hydrodynamical description a plane plasma wave produces both a nonlinear current $e\overline{n_1 \mathbf{v}_1}$ and also an oppositely directed nonlinear current $en_0\overline{\mathbf{v}_2}$ exactly annihilating the first one. Here the quantities n_1, $\mathbf{v}_1$, and $\mathbf{v}_2$ are the electron density and velocity perturbations in a plasma wave of the first and second order, respectively, and the overbar means temporal averaging. This annihilation occurs also in a strong nonlinear plane plasma wave, where only the relativistic nonlinearity is present.

The plasma wake considered here, however, is a radially limited plasma wave and may be interpreted as a wave beam rather than a plane plasma wave. In this case the annihilation of the averaged nonlinear currents is no longer local but occurs over the total beam cross section.

To support and illustrate this statement we used Ampère's equation and found the current density profile that is responsible for the homogeneous part of the magnetic field generated by a wide laser pulse. In accordance with Eq. (72), we obtained

$$j_z = -j_0(1-2\alpha^2\rho^2)\exp(-2\alpha^2\rho^2), \tag{73}$$

where

$$j_0 = (en_0 c\alpha^2 a_0^4/4)\sin^2(k_p L/2)[(k_p/k_L)^2 - 1]^{-2}$$

is the current density on the wake axis. Figure 4 shows the radial variation of normalized current density as well as the normalized laser intensity $|a|^2$. It is seen that in the vicinity of the pulse axis the current is negative. This implies that the electrons flow in the direction of pulse propagation. However, in the edge of the pulse the current is positive and the electrons move in the opposite direction. The total steady current in the cross section of the plasma wake is zero.

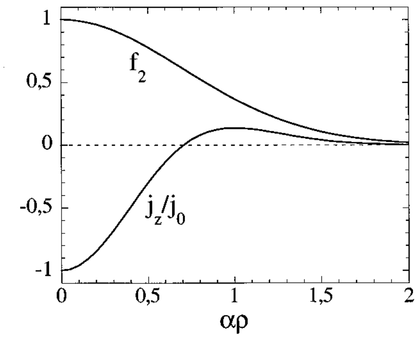

FIG. 4. The normalized steady current density j_z/j_0 and pulse intensity $f_2(\rho)$ versus dimensionless radial variable ρ.

The magnetic field considered here is able to influence the movement of accelerated electrons in the laser wakefield accelerator,[17,22,23] especially in the case of intense laser pulses when the normalized amplitude a_0 exceeds unity. Though the study of electron trajectories in the electric and magnetic fields of the wake of an intense laser pulse is beyond the scope of the present paper, simple arguments show that the magnetic field has a beneficial effect on the accelerated electrons. As noted above, the azimuthal magnetic field is always negative. Therefore it has a focusing effect on relativistic electrons injected in the wakefield, which feel the pinch force due to the nonlinear current of the plasma wave. In the case of Fig. 3(b), the homogeneous normalized magnetic field is of the order of -0.07 in the wake while the normalized radial electric field $(eE_r/mc\omega_p)$ of the plasma wake oscillates between -0.19 and 0.17. As a positive radial electric field and a negative azimuthal magnetic field focus on relativistic electrons, we conclude that the self-generated magnetic field significantly extends the focusing part of the plasma wake.

ACKNOWLEDGMENTS

One of the authors (L.M.G.) would like to thank the French Minister of Education, Universities and Research for an invitation to work in France, the University Paris-Sud in Orsay, for their hospitality, and especially Professor G. Mattieussent for providing his visit in France.

This work was supported in part by the Russian Foundation for Basic Research (Grant No. 94-02-03847), in part by INTAS (Contract No. 94-0870).

APPENDIX A: CALCULATION OF j_1

Here we calculate that part of the quasisteady current $\mathbf{j}_1$ that appears in Eq. (27) due to second-harmonic terms. Using Eq. (34) we find that Eq. (33) becomes

$$\phi_2 = -\frac{\omega_p^2}{16\omega^2}\left\{\left[a^2\left(1+\frac{\omega_p^2}{4\omega^2}\right)-\frac{i}{\omega}\frac{\partial a^2}{\partial t} - \frac{3}{4\omega^2}\frac{\partial^2 a^2}{\partial t^2}\right]e^{-2i\omega t+2ikz}+\text{c.c.}\right\}. \quad \text{(A1)}$$

Substituting (A1) into the expression $\Delta\phi_2$ we obtain, by means of (35)–(36),

$$\Delta\phi_2 = -\frac{\omega_p^2}{16\omega^2}\left\{\left[4k^2a^2\left(1+\frac{\omega_p^2}{4\omega^2}\right)-4ik^2\left(\frac{1}{\omega}\frac{\partial a^2}{\partial t}+\frac{1}{k}\frac{\partial a^2}{\partial z}\right) + 4\frac{k}{\omega}\frac{\partial^2 a^2}{\partial t\partial z}-\frac{3k^2}{\omega^2}\frac{\partial^2 a^2}{\partial t^2}-\Delta a^2\right] \times e^{-2i\omega t+2ikz}+\text{c.c.}\right\}. \quad \text{(A2)}$$

From Eq. (30) and the dispersion law $k^2c^2=\omega^2-\omega_p^2$, we obtain

$$\frac{1}{\omega}\frac{\partial a^2}{\partial t}+\frac{1}{k}\frac{\partial a^2}{\partial z}=\frac{1}{k}\frac{\omega_p^2}{\omega^2}\frac{\partial a^2}{\partial z}. \quad \text{(A3)}$$

Substituting Eqs. (31) and (A2) into the second harmonic part of the form $q_1^2-c^2\,\Delta\phi/\omega_p^2$ involved in Eq. (27), which we denote S_2, we find

$$S_2=\left[\frac{a^2}{4}\frac{\omega_p^2}{\omega^2}\left(1-\frac{k^2c^2}{4\omega^2}\right)+\frac{3c^2k^2}{16\omega^4}\frac{\partial^2 a^2}{\partial t^2}+\frac{c^2k}{4\omega^3}\frac{\partial^2 a^2}{\partial t\,\partial z} + \frac{c^2}{16\omega^2}\Delta a^2\right]e^{-2i\omega t+2ikz}+\text{c.c.} \quad \text{(A4)}$$

It is important to emphasize that in S_2 the zeroth-order terms are reduced and consequently expression (A4) contains only second-order terms.

Equation (A4) may be simplified also by means of the equation

$$\frac{\partial a^2}{\partial t}+v_g\frac{\partial a^2}{\partial z}=0, \quad \text{(A5)}$$

which is the consequence of Eq. (30). As a result of its application, we find

$$S_2=(3\omega_p^2a^2+c^2\,\Delta_\perp a^2)\frac{1}{16\omega^2}e^{-2i\omega t+2ikz}+\text{c.c.}, \quad \text{(A6)}$$

where $\Delta_\perp$ is the transverse part of the Laplace operator.

Using Eqs. (A1), (A6), and (35)–(36), we obtain that the contribution of the second-harmonic terms in quasisteady current, which we denote as $\overline{\mathbf{j}_{12}}$, has the form

$$\overline{\mathbf{j}_{12}}=-\frac{3ecn_0}{128}\frac{\omega_p^2}{\omega^2}\mathbf{e_z}\left[|a|^4+\frac{1}{6}\frac{c^2}{\omega_p^2}(a^2\Delta_\perp a^{*2} + a^{*2}\Delta_\perp a^2)\right], \quad \text{(A7)}$$

where $\mathbf{e_z}$ is the axial unit vector. It is seen that the current (A7) contains the small factor $(\omega_p/\omega)^2$ as compared with the current (37).

APPENDIX B: CALCULATION OF j_2

We consider here the quasisteady component of the current $\mathbf{j_2}$, denoting it as $\overline{\mathbf{j}_2}$. Equation (28) contains the general factor $\mathbf{q_1}$ determined by the expression (29). It is evident that the zero harmonic in the current arises only from those terms in the curved brackets of Eq. (28), which contain also the multipliers $e^{\pm(i\omega t-ikz)}$. To take into account such terms we first find the quantity $\mathbf{q_2}$. Using the definition (26) and Eqs. (32) and (A1), we obtain

$$\mathbf{q_2}=\mathbf{q}_{20}+(\mathbf{q}_{22}e^{-2i\omega t+ikz}+\text{c.c.}) \quad \text{(B1)}$$

where

$$\mathbf{q}_{20}=-\frac{c}{4}\nabla\int_{-\infty}^{t}dt\,\cos[\omega_p(t-t')]|a(t',\mathbf{r})|^2, \quad \text{(B2)}$$

$$\mathbf{q}_{22}=\frac{c}{16\omega^2}\left[2\omega k\mathbf{e_z}a^2-ik\mathbf{e_z}\frac{\partial a^2}{\partial t}-i\omega\,\nabla a^2 - \frac{k\mathbf{e_z}}{2\omega}\left(\frac{\partial^2 a^2}{\partial t^2}-\omega_p^2a^2\right)-\frac{1}{2}\frac{\partial}{\partial t}\nabla a^2\right]. \quad \text{(B3)}$$

In accordance with Eqs. (B1)–(B3) and (29) the first harmonic term in the product $\mathbf{q_1}\cdot\mathbf{q_2}$ has the form

$$(\mathbf{q_1}\cdot\mathbf{q_2})_1=be^{-\omega t+iz}+b^*e^{i\omega t+ikz}, \quad \text{(B4)}$$

where the bottom index indicates the number of the harmonic, and

$$b=\tfrac{1}{2}(\mathbf{a}\cdot\mathbf{q}_{20}+\mathbf{a}^*\cdot\mathbf{q}_{22}). \quad \text{(B5)}$$

By means of Eqs. (34), (35)–(36), and (B4), we find

$$\Delta\int_{-\infty}^{t}dt'\,\sin[\omega_p(t-t')][\mathbf{q}_1(t')\cdot\mathbf{q}_2(t')]_1 = -\frac{\omega_p}{\omega^2}\left\{\left[-k^2b+2ik^2\left(\frac{1}{\omega}\frac{\partial b}{\partial t}+\frac{1}{k}\frac{\partial b}{\partial z}\right)+\Delta b + \frac{4k}{\omega}\frac{\partial^2 b}{\partial t\,\partial z}-\frac{k^2}{\omega^2}\left(\omega_p^2b-3\frac{\partial^2 b}{\partial t^2}\right)\right]e^{-i\omega t+ikz}+\text{c.c.}\right\}. \quad \text{(B6)}$$

We consider the first-order term in Eq. (B6). According to (B5), we have

$$\frac{1}{k}\frac{\partial b}{\partial z}+\frac{1}{\omega}\frac{\partial b}{\partial t}=\frac{1}{2}\left\{\mathbf{q}_{20}\cdot\left(\frac{1}{k}\frac{\partial \mathbf{a}}{\partial z}+\frac{1}{\omega}\frac{\partial \mathbf{a}}{\partial t}\right)\right]+\mathbf{q}_{22}\cdot\left(\frac{1}{k}\frac{\partial \mathbf{a}^*}{\partial z}+\frac{1}{\omega}\frac{\partial \mathbf{a}^*}{\partial t}\right)+\mathbf{a}\cdot\left(\frac{1}{k}\frac{\partial \mathbf{q}_{20}}{\partial z}+\frac{1}{\omega}\frac{\partial \mathbf{q}_{20}}{\partial z}\right)\right]+\mathbf{a}^*\cdot\left(\frac{1}{k}\frac{\partial \mathbf{q}_{22}}{\partial z}+\frac{1}{\omega}\frac{\partial \mathbf{q}_{22}}{\partial z}\right)\right\}. \tag{B7}$$

Using Eq. (30) and the dispersion law, we obtain

$$\frac{1}{k}\frac{\partial \mathbf{a}}{\partial z}+\frac{1}{\omega}\frac{\partial \mathbf{a}}{\partial z}=\frac{1}{k}\frac{\omega_p^2}{\omega^2}\frac{\partial \mathbf{a}}{\partial z}. \tag{B8}$$

This means that the first two terms in (B7) contain the additional small factor (ω_p^2/ω^2) and may be omitted as terms of third order. The third term in (B7) also contains the small factor (ω_p^2/ω^2). Really, we have, in accordance with (B2) and (A3),

$$\frac{1}{k}\frac{\partial \mathbf{q}_{20}}{\partial z}+\frac{1}{\omega}\frac{\partial \mathbf{q}_{20}}{\partial t}=-\frac{c}{4}\,\boldsymbol{\nabla}\int_{-\infty}^{t}dt'\ \cos[\omega_p(t-t')]\times\left(\frac{\partial}{k\partial z}+\frac{\partial}{\omega\partial t'}\right)a^2(t',\mathbf{r}). \tag{B9}$$

And finally, the last term in (B7), as it is seen from Eqs. (B3) and (B6), receives also the additional small parameter $(\omega_p/\omega)^2$. Then, the leading-order term in Eq. (B6) is, in reality, a term of the third order, and it is permissible to omit it.

Using Eqs. (B4) and (B6), we find for the difference arising from the first and second terms in the curved brackets of Eq. (28),

$$(\mathbf{q}_1\cdot\mathbf{q}_2)_1-\frac{c^2}{\omega_p}\,\Delta\int_{-\infty}^{t}dt'\ \sin[\omega_p(t-t')][\mathbf{q}_1(t')\cdot\mathbf{q}_2(t')]_1=b\left(1-\frac{k^2c^2}{\omega^2}\right)+\frac{c^2}{\omega^2}\,\Delta b+\frac{4kc^2}{\omega^3}\frac{\partial^2 b}{\partial z\ \partial t}-\frac{c^2k^2}{\omega^2}\frac{\omega_p^2}{\omega^2}\,b+3\,\frac{c^2k^2}{\omega^4}\frac{\partial^2 b}{\partial t^2}. \tag{B10}$$

It is seen that the main zeroth-order terms in (B10) are reduced, producing a term of second order. This circumstance permits us to consider in expression (B5) for b only leading-order terms,

$$b=\frac{1}{16}\left(-2c\mathbf{a}\boldsymbol{\nabla}\int_{-\infty}^{t}dt'\ \cos[\omega_p(t-t')]|a(t')|^2+\frac{kc}{\omega}\,a^2a_z^*\right). \tag{B11}$$

The condition $\boldsymbol{\nabla}\cdot\mathbf{q}_1=0$ implies, according to (29), $ika_z=-\boldsymbol{\nabla}_\perp\cdot\mathbf{a}$. So, both terms in (B11) are of first order with respect to parameter $k^{-1}\boldsymbol{\nabla}_\perp$ and Eq. (A10) contains only terms of third order and can be ignored.

The last term in Eq. (28) has the multiplier considered earlier,

$$S=q_1^2-\frac{c^2}{\omega_p^2}\,\Delta\phi=S_0+(S_2e^{-2i\omega t+2ikz}+\text{c.c.}), \tag{B12}$$

where $S_0=|a^2|/2-c^2\ \Delta\phi_0/\omega_p^2$ and the values ϕ_2 and S_2 are determined by Eqs. (32) and (A4), respectively. Using (B12) we obtain the first harmonic terms in the expression $\mathbf{q}_1\cdot\boldsymbol{\nabla}S$ and perform the temporal integration according to (34). Multiplying the result by $\mathbf{q}_1$, we find the zero harmonic. The main contribution in the current (28) appears in the form

$$\overline{\mathbf{J}_2}=\frac{c^2en_0}{8\omega^2}\left(\mathbf{a}^*\,\frac{\partial}{\partial \mathbf{t}}\,(\mathbf{a}\cdot\boldsymbol{\nabla}S_0)+\mathbf{a}\,\frac{\partial}{\partial t}\,(\mathbf{a}^*\cdot\boldsymbol{\nabla}S_0)\right). \tag{B13}$$

APPENDIX C: CALCULATION OF $G_{1,2}$ FOR $\alpha<1$

In the general case, by substituting the functions (62) and (63) into Eq. (55), we obtain two kinds of integrals,

$$J_n=I_1(\rho)\int_{\rho}^{\infty}dx\ x^{2n}K_1(x)e^{-2\alpha^2x^2}+K_1(\rho)\int_0^{\rho}dx\ x^{2n}I_1(x)e^{-2\alpha^2x^2}, \tag{C1}$$

where $n=1$ or 2. The integral J_2 is expressed through the integral J_1 by means of the operation $J_2=-\partial J_1/\partial(2\alpha^2)$. Hence, for all calculations it is sufficient to know the integral J_1. In the case of a wide pulse ($\alpha^2<1$), it is convenient to represent the integral J_1 in the form of a power series in α^2. It may be done as a result of repeated integration by parts,

$$\int_{\rho}^{\infty}dx\ x^2K_1(x)e^{-2\alpha^2x^2}=\sum_{n=0}^{\infty}(-4\alpha^2)^n\rho^{2+n}K_{2+n}(\rho)e^{-2\alpha^2\rho^2}, \tag{C2}$$

and

$$\int_0^{\rho}dx\ x^2I_1(x)e^{-2\alpha^2x^2}=\sum_{n=0}^{\infty}(4\alpha^2)^n\rho^{2+n}I_{2+n}(\rho)e^{-2\alpha^2\rho^2}. \tag{C3}$$

Substituting (C2) and (C3) into (C1), we find

$$J_1=\sum_{n=0}^{\infty}(4\alpha^2)^n\rho^{2+n}e^{-2\alpha^2\rho^2}[(-1)^nI_1(\rho)K_{2+n}(\rho)+K_1(\rho)I_{2+n}(\rho)]. \tag{C4}$$

The recursion relations for the Bessel functions,[18]

$$K_{\nu+1}=K_{\nu-1}+\frac{2\nu}{\rho}\,K_\nu,\quad I_{\nu+1}=I_{\nu-1}-\frac{2\nu}{\rho}\,I_\nu, \tag{C5}$$

permit one to compute all the terms in the square brackets of Eq. (C4) knowing the first one,

$$I_1(\rho)K_2(\rho)+K_1(\rho)I_2(\rho)=\frac{1}{\rho}. \tag{C6}$$

Limiting expansion (C4) to the two first leading terms, one finds

$$J_1=\rho e^{-2\alpha^2\rho^2}[1-16\alpha^2(1-\alpha^2\rho^2)]. \tag{C7}$$

Inserting Eqs. (62) and (63) in the integrals (55) and using Eq. (C7), we obtain Eq. (70).

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